Death process - Answers

$P[N(t+h) = n-k|N(t) = n] = \begin{cases} 1-\mu_n h + o(h), & k = 0 \\ \mu_n h + o(h), & \text{k=1} \\ o(h), & k \geq 2. \end{cases}$

where μ_n is the rate at which the births occur at time t and n being the size of the population at time t.

Question 1

Suppose that a population has an average death rate of μ_n. Let $P_n(t)$ be the probability that there are n individuals in the population at time t. Assume that initially, there are n_0 number of individuals at time $t = 0$. Derive the following system of differential equations for $P_n(t)$.

$P'_{n_0}(t) = -\mu_n P_n(t)$ and

$P'_n(t) = -\mu_n P_n(t) + \mu_{n+1} P_{n+1}(t)$ for $0 \leq n < n_0$.

Note: These system of differential equations can be solved subject to the conditions $P_{n_0}(0) = 1$ and $P_n(0) = 0$ for $0 \leq n < n_0$.

Hint: You can obtain a system of differential equations similar to the pure birth process.

Q1 - Answer

Let us consider a death process whose total number of individuals at time t is denoted by a discrete random variable $N(t)$. Furthermore, $\{N(t) : t \geq 0\}$ represents a stochastic process with a continuous parameter space and a discrete state space. Assume that initially, there are n_0 number of individuals at time $t = 0$.

Let $P_n(t) = P[N(t) = n]$

Then, for $0 \leq n < n_0$,

$$\begin{aligned} P_n(t+h) = P(N(t) = n)P(N(t+h) = n|N(t) = n) + \\ P(N(t) = n+1)P(N(t+h) = n|N(t) = n+1) + \\ \sum_{r=2}^{\infty} P(N(t) = n+r)P(N(t+h) = n|N(t) = n+r) \end{aligned}$$

i.e

$$\begin{aligned} P_n(t+h) = P_n(t)(1-\mu_n h + o(h)) + \\ P_{n+1}(t)(\mu_{n+1} h + o(h)) + \\ o(h) \end{aligned}$$

i.e

$$P_n(t+h) = P_n(t) - \mu_n P_n(t)h + P_{n+1}(t)\mu_{n+1} h + o(h) \text{ for } 0 \leq n < n_0.$$

$$lim_{h\to 0} \frac{P_n(t+h) - P_n(t)}{h} = -\mu_n P_n(t) + \mu_{n+1} P_{n+1}(t) + lim_{h\to 0} \frac{o(h)}{h}.$$

i.e.

$$P_n'(t) = -\mu_n P_n(t) + \mu_{n+1} P_{n+1}(t) \text{ for } 0 \leq n < n_0.$$

For $n = n_0$

$$P_{n_o}(t+h) = P[N(t) = n_0]P[N(t+h) = 0|N(t) = n_0],$$

$$P_{n_o}(t+h) = P_{n_0}(t)[1 - \mu_{n_0}h + o(h)],$$

$$P_{n_o}(t+h) = P_{n_0}(t) - \mu_{n_0}hP_{n_0}(t) + o(h),$$

$$lim_{h\to 0}\frac{P_{n_0}(t+h) - P_{n_0}(t)}{h} = -\mu_{n_0}P_{n_0}(t) + lim_{h\to 0}\frac{o(h)}{h}.$$

$$P_{n_0}' = -\mu_{n_0}P_{n_0}(t).$$

Question 2: Linear Death Process - PMF

When $\mu_n = n\mu$, i.e. when the death rate is linear in the present size of the population, the pure death process is said to be a **linear death process**. Let us assume that there are n_0 individuals in the population initially.

i) When $\mu_n = n\mu$, obtain the system of differential equations of the linear death process.

When $\mu_{n_0} = n_0\mu$,

$$P_{n_0}' = -\mu_{n_0}P_{n_0}(t),$$

becomes

$$P_{n_0}' = -n_0\mu P_{n_0}(t).$$

Furthermore, when $\mu_n = n\mu$, $P_n'(t) = -\mu_n P_n(t) + \mu_{n+1}P_{n+1}(t)$ for $0 \leq n < n_0$ becomes

$$P_n'(t) = -n\mu P_n(t) + (n+1)\mu P_{n+1}(t) \text{ for } 0 \leq n < n_0,$$

with initial conditions $P_{n_0}(0) = 1$ and $P_n(0) = 0$ for $0 \leq n < n_0$.

Q2-i: Answer

When $\mu_n = n\mu$, i.e. when the death rate is linear in the present size of the population, the pure death process is said to be a **linear death process**.

ii) Based on the system of differential equations show that

$$P_n(t) = \frac{n_0!}{(n_0 - n)!n!}(e^{-\mu t})^n(1 - e^{-\mu t})^{n_0 - n},$$

for $0 \leq n \leq n_0$.

Q2-ii: Answer

$$P_n'(t) = -n\mu P_n(t) + (n+1)\mu P_{n+1}(t) \text{ for } 0 \le n < n_0,$$

Multiplying the equation for n by z^n and summing over all n we obtain

$$\frac{\partial}{\partial t}\sum_{n=1}^{n_0} P_n(t)z^n = -\mu z\frac{\partial}{\partial z}\sum_{n=1}^{n_0} P_n(t)z^n + \mu\frac{\partial}{\partial z}\sum_{n=1}^{n_0-1} P_{n+1}(t)z^{n+1}$$

Let $\prod(z,t) = \sum_{n=0}^{n_0} P_n(t)z^n$. Then the above equations becomes

$$\frac{\partial\prod(z,t)}{\partial t} = -\mu z\frac{\partial\prod(z,t)}{\partial z} + \mu\frac{\partial\prod(z,t)}{\partial z}$$

i.e.

$$\frac{\partial\prod(z,t)}{\partial t} = -\mu(z-1)\frac{\partial\prod(z,t)}{\partial z}$$

Subsidiary equations take the form

$$\frac{dt}{1} = \frac{dz}{\mu(z-1)} = \frac{d\prod}{0}$$

Two independent solutions can be obtained one from $d\prod = 0$ and the other from $\mu dt = \frac{dz}{(z-1)}$.

$$d\prod = 0 \Rightarrow \prod(z,t) = constant.$$

$$-\mu dt = \frac{dz}{(z-1)} \Rightarrow (z-1)e^{-\mu t} = constant.$$

The general solution can be written as

$\prod(z,t) = f\left((z-1)e^{-\mu t}\right)$ where f is an arbitrary function.

The initial conditions $P_{n_0}(0) = 1$ and $P_n(0) = 0$ for $0 \le n < n_0$ imply that $\prod(z,0) = z^i$.

$$\therefore \prod(z,0) = f(z-1) = z^i.$$

Let $\omega = (z-1) \Rightarrow z = \omega + 1$ and hence we obtain $f(\omega) = (\omega+1)^i$.

$$\therefore \prod(z,t) = f(z-1) = z^i.$$

$$\therefore \prod(z,t) = \left((z-1)e^{-\mu t} + 1\right)^i = \{ze^{-\mu t} + (1-e^{-\mu t})\}^i.$$

In fact this is the probability generating function of a Binomial distribution with parameter $e^{-\mu t}$ and i.

Considering coefficient of z^n we have

$$P_n(t) = \frac{n_0!}{(n_0-n)!n!}(e^{-\mu t})^n(1-e^{-\mu t})^{n_0-n},$$

Question 3: The mean and variance of the pure death process

Show that the mean of the pure linear death process is

$E(N(t)) = n_0 e^{-\mu t}$

and the variance is

$V(N(t)) = n_0 e^{-\mu t}(1 - e^{-\mu t})$.

Q3 - Answer

Prove if $X \sim Bin(n, p)$, then $E(X) = np$ and $V(X) = npq$. Here, $p = e^{-\mu t}$ and $q = (1 - e^{-\mu t})$.

Hence,

$E(N(t)) = n_0 e^{-\mu t}$

and the variance is

$V(N(t)) = n_0 e^{-\mu t}(1 - e^{-\mu t})$.

Question 4: Extinction

In the pure death process the population either remains constant or it decreases. It may eventually reach zero in which case we say that the population has gone **extinct**. Show that the probability the population is extinct at time t is given by

$$P(N(t) = 0 | N(0) = n_0) = (1 - e^{-\mu t})^{n_0}.$$

Q4 - Answer

$$P(N(t) = 0 | N(0) = n_0) = \frac{n_0!}{(n_0 - 0)! n 0!}(1 - e^{-\mu t})^{n_0}.$$

Hence,

$$P(N(t) = 0 | N(0) = n_0) = (1 - e^{-\mu t})^{n_0}.$$