STA 331 2.0 Stochastic Processes

11. Birth-and-Death Process - important results (cont)

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Definition

A continuous parameter stationary Markov process is a stochastic process having the properties that

1. Each time it enters state i, the amount of time it spends in that state before making a transition into a different state is exponentially distributed (say with rate ν_i or mean $\frac{1}{\nu_i}$), and
2. When the process leaves state i, it enters state j with some probability, p_{ij} satisfying,

$$P_{ii} = 0 \text{ all } i$$

$$\sum_j P_{ij} = 1 \text{ all } i$$

Birth-and-death process

For birth and death process, let λ_i and μ_i be given by

$\lambda_i = q_{i,i+1}$ and $\mu_i = q_{i,i-1}$.

The values $\{\lambda_i, i \geq 0\}$ and $\{\mu_i, i \geq 0\}$ are called respectively the birth and death rate. Then

$$\nu_i = \lambda_i + \mu_i$$

Then $T_i \sim exp(\lambda_i + \mu_i)$.

Furthermore,

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i} = 1 - P_{i,i-1}$$

Examples of birth-and-death process

Example 1:

Suppose that life-time of a component of a machine is exponentially distributed with rate λ.

Let $X(t)$ be the state of the machine at time t.

$$X(t) = \begin{cases} 1, & \text{if the machine is operational at time t} \\ 0, & \text{if the machine is not operational at time t} \end{cases} \tag{1}$$

This is a continuous parameter discrete state Markov process with absorbing barrier state 0 (suppose that there are no repairs).

Examples of birth-and-death process

Example 1 (cont.):

Instantaneous transition probabilities

$$P_{01} = 0$$

and

$$P_{10} = 1$$

.

Hence, $P_{11} + P_{10} = 1$.

Examples of birth-and-death process

Example 1 (cont.):

The probability mass function

$$P_1(t) = P(X(t) = 1) = ?$$
$$P_0(t) = P(X(t) = 0) = ?$$

You know the distribution of T_1. Hence,

$$[X(t) = 1] \Rightarrow [T_1 \geq t].$$

Therefore,

$$P(X(t) = 1) = P(T_1 \geq t) = \int_t^{\infty} \lambda e^{-\lambda u} du.$$

Examples of birth-and-death process

Example 1 (cont.):

Your turn: calculate the associate probabilities for,

$$P_1(t) = P(X(t) = 1) = ?$$

$$P_0(t) = P(X(t) = 0) = ?$$

Examples of birth-and-death process

Example 1 (cont.):

second type of transition probability $P_{ij}(t)$.

$$P_{10}(t) = P(X(t+s) = 0 | X(s) = 1).$$

Write this interms of T_1

$$P_{10}(t) = P(T_1 \leq t).$$

$T_1 \sim exp(\lambda)$

You can show that $P_{10}(t) = 1 - e^{-\lambda t}$ and

$P_{11}(t) = e^{-\lambda t}$.

Examples of birth-and-death process

Example 2:

A machine is operational for a time that is exponentially distributed with rate α and off or down for a time that is exponentially distributed with rate β. For example, the machine needs a part that has an exponentially distributed lifetime; once it burns out, the fix-it time (time required to obtain a new part) is also exponentially distributed.

$$X(t) = \begin{cases} 1, & \text{if the machine is operational at time t} \\ 0, & \text{if the machine is not operational at time t} \end{cases} \tag{2}$$

This is a continuous parameter discrete state Markov process
There is no absorbing barrier state.

Examples of birth-and-death process

Example 2:

Instantaneous transition prob.	PMF	Transition Prob.
P_{01}	$P_1(t)$	$P_{00}(t)$ $P_{01}(t)$
P_{01}	$P_0(t)$	$P_{10}(t)$ $P_{11}(t)$

$T_1 \sim exp(\alpha)$ and $T_0 \sim exp(\beta)$.

T_0 - time it takes to repair the component.

Examples of birth-and-death process

Example 3 (One server queue)

Suppose that there is **one** checkout counter at a shopping store. Suppose that customers arrive to a single server system according to a Poisson processing having rate λ and the service time is exponentially distributed with parameter ν.

Examples of birth-and-death process

Example 4 (Two server queue)

Suppose that there are **two** parallel identical checkout counter at a shopping store. The service times are independently and identically distributed. Suppose that customers arrive according to a Poisson processing having rate λ and the service time is exponentially distributed with parameter ν.

Examples of birth-and-death process

Example 5

A telephone operator has a phone with a hold button. Suppose incoming voice calls arrive according to a Poisson process with rate λ. Each call takes an exponentially distributed time with average $1/\nu$ minutes. If a call arrives during a time the phone is busy, it is placed on hold. If another call arrives, it receives a busy tone and must hang up.