

# STA 331 2.0 Stochastic Processes

## 8. Pure Death Process

---

Dr Thiyanga S. Talagala

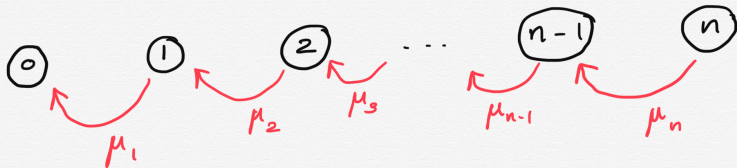
Department of Statistics, University of Sri Jayewardenepura

# Pure Death Process

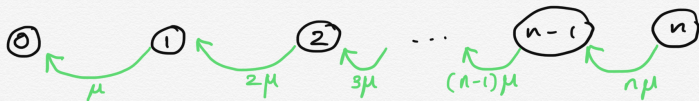
- Individuals persist only until they die and there are no births.
- We assume initially, there are  $n_0$  number of individuals at time  $t = 0$ .

# The state transition diagram.

Pure death process



Linear death process (when  $\mu_n = n\mu$ )



# Pure Death Process

Let us consider a death process whose total number of individuals at time  $t$  is denoted by a discrete random variable  $N(t)$ . As parameter  $t$  varies  $\{N(t) : t \geq 0\}$  represent a stochastic process with a continuous parameter space and a discrete state space.

We assume that the individuals of a death process with initial size  $n_0$  die at a certain rate, which depends on the present size of the population eventually reducing the size to the zero.

# Pure Death Process

## Condition 1

$$P[N(t+h) = n-k | N(t) = n] = \begin{cases} 1 - \mu_n h + o(h), & k=0 \\ \mu_n h + o(h), & k=1 \\ o(h), & k \geq 2. \end{cases}$$

where  $\mu_n$  is the rate at which the births occur at time  $t$  and  $n$  being the size of the population at time  $t$ .

## Condition 2

$$N(0) = n_0$$

# Notation

Let  $N(t)$  be the number of individuals alive at time  $t$ . Suppose initially, there are  $n_0$  individuals, that is  $N(0) = n_0$ .

$$P_n(t) = P[N(t) = n]$$

# Linear Death Process

Suppose  $\mu_n = n\mu$ , and initially,  $N(0) = n_0$ .

Then we can show that,

$$N(t) \sim \text{Binomial}(n_0, p)$$

where  $p = e^{-\mu t}$ . That is,

$$P_n(t) = \frac{n_0!}{(n_0 - n)!n!} (e^{-\mu t})^n (1 - e^{-\mu t})^{n_0 - n},$$

for  $0 \leq n \leq n_0$ .

## Question 1

Suppose that a population has an average death rate of  $\mu_n$ . Let  $P_n(t)$  be the probability that there are  $n$  individuals in the population at time  $t$ . Assume that initially, there are  $n_0$  number of individuals at time  $t = 0$ . Derive the following system of differential equations for  $P_n(t)$ .

$$P'_{n_0}(t) = -\mu_{n_0}P_{n_0}(t) \text{ and}$$

$$P'_n(t) = -\mu_n P_n(t) + \mu_{n+1} P_{n+1}(t) \text{ for } 0 \leq n < n_0.$$

Note: These system of differential equations can be solved subject to the conditions  $P_{n_0}(0) = 1$  and  $P_n(0) = 0$  for  $0 \leq n < n_0$ .

Hint: You can obtain a system of differential equations similar to the pure birth process.



## Question 2: Linear Death Process - PMF

When  $\mu_n = n\mu$ , i.e. when the death rate is linear in the present size of the population, the pure death process is said to be a **linear death process**. Let us assume that there are  $n_0$  individuals in the population initially.

- i) When  $\mu_n = n\mu$ , obtain the system of differential equations of the linear death process.
- ii) Based on the system of differential equations show that

$$P_n(t) = \frac{n_0!}{(n_0 - n)!n!} (e^{-\mu t})^n (1 - e^{-\mu t})^{n_0 - n},$$

for  $0 \leq n \leq n_0$ .

## Question 3: The mean and variance of the pure death process

Show that the mean of the pure linear death process is

$$E(N(t)) = n_0 e^{-\mu t}$$

and the variance is

$$V(N(t)) = n_0 e^{-\mu t} (1 - e^{-\mu t}).$$

## Question 4: Extinction

In the pure death process the population either remains constant or it decreases. It may eventually reach zero in which case we say that the population has gone **extinct**. Show that the probability the population is extinct at time  $t$  is given by

$$P(N(t) = 0 | N(0) = n_0) = (1 - e^{-\mu t})^{n_0}.$$