STA 331 2.0 Stochastic Processes*

Birth-and-Death Processes

Prerequisite: Partial Differential Equations

A partial differential equation for a function $z(x, y)$ is Lagrange type if it takes the form

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z). \tag{1}$$

This equation contains i) partial differential coefficients, ii) independent variables and iii) dependent variables.

The auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{2}$$

Let $u(x,y,z) = C_1$ and $v(x,y,z) = C_2$ are two independent solutions of auxiliary equations. Here, C_1 and C_2 are constants.

Then the general solution to equation (1) can be written as $f(u,v) = 0$ or $u = \phi(v)$.

Example 1

Solve the following partial differential equation,

$$y^2p - xyq = x(z - 2y),$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}. \tag{3}$$

Consider the first two terms in the equation (3)

$$\frac{dx}{y} = \frac{dy}{-x} \tag{4}$$

*Thiyanga S Talagala

$$xdx = -ydy. \tag{5}$$

Integrating equation (5) we get

$$\frac{x^2}{2} = -\frac{y^2}{2} + \frac{C_1}{2}.$$

Hence, we have

$$x^2 + y^2 = C_1. \tag{6}$$

Consider the last two terms in the equation (3)

$$\frac{-dy}{xy} = \frac{dz}{x(z-2y)} \tag{7}$$

$$2ydy = ydz + zdy. \tag{8}$$

Integrating equation (7) we get

$$y^2 - yz = C_2. \tag{9}$$

From equations (6) and (9)

$$x^2 + y^2 = f(y^2 - yz).$$

Answer: $x^2 + y^2 = f(y^2 - yz)$.

Example 2 (This is related to the proof of birth process).

Solve the following partial differential equation.

$$\frac{\partial \prod(z,t)}{\partial t} = -\lambda z \frac{\partial \prod(z,t)}{\partial z} + \lambda z^2 \frac{\partial \prod(z,t)}{\partial z} \tag{10}$$

where

$$\prod(z,t) = \sum_{n=1}^{\infty} P_n(t) z^n.$$

We can simplify the equation (10) and obtain

$$\frac{\partial \prod(z,t)}{\partial t} = \lambda z(z-1) \frac{\partial \prod(z,t)}{\partial z} \tag{11}$$

Auxiliary/Subsidiary equations take the form

$$\frac{dt}{1} = \frac{dz}{-\lambda z(z-1)} = \frac{d\prod}{0}. \tag{12}$$

From the first two parts of the equation (12)

$$-\lambda dt = \frac{dz}{z(z-1)} \tag{13}$$

From the last two parts of the equation (12)

$$d\prod = 0. \tag{14}$$

Integrating equations (13) and (14)

$$\frac{z}{z-1}e^{-\lambda t} = constant \tag{15}$$

$$\prod(z,t) = constant \tag{16}$$

The general solution can be written as,

$$\prod(z,t) = f\left(\frac{z}{z-1}e^{-\lambda t}\right)$$

where f is an arbitrary function.